POLYMATH JR 2025

FINDING ELLIPSES (GROUP)

Contents

1. Introduction

2. Decomposable Blaschke Products

2.1. Original approach to Blaschke products conjugate to z^n.

2.1.1. *Definitions.* Let $n \in \mathbb{N}$ be fixed throughout.

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$
$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$
$$g : z \mapsto z^n$$
$$\varphi_a^\mu : z \mapsto \mu \frac{z-a}{1-\bar{a}z} \text{ for } \mu \in \mathbb{T}, a \in \mathbb{D}$$

2.2. **Background.** I seek to describe the Blaschke curves for Blaschke products of the form $B = \varphi_{a'}^{\mu'} \circ g \circ \varphi_a^\mu$. I claim that they are always ellipses.

Without loss of generality, I will consider the case $B(z) = g \circ \varphi_a^\mu(z)$.

I will rely on the following result from *Marc Frantz (2004) How Conics Govern Möbius Transformations, The American Mathematical Monthly.* I have adapted it slightly, to fit the notation of the Finding ellipses textbook, and consider only the case where $|a| < 1$, so the general conic is an ellipse.

Theorem 1. *Choose $\mu \in \mathbb{T}, a \in \mathbb{D}$.*

Let C_a^μ be the ellipse with foci $a, -\mu a$ and eccentricity $|a| = |\mu a|$.

Then there is a one-to-one correspondence between tangents to C_a^μ and lines connecting λ and $\phi_a^\mu(\lambda)$ for λ on the unit circle,

2.2.1. *Proof.* **Notation:** Let ω_n be the principal n'th root of unity, and let $\sqrt[n]{x}$ be the principal n'th root of x.

Choose λ on the unit circle. Then it's preimage is

$$B^{-1}(\lambda) = \{z_i : i = 1, 2 \cdots n\} = \{(\varphi_a^\mu)^{-1}\left(\omega_n^i \sqrt[n]{\lambda}\right) : i = 1, 2, \cdots n\}$$

The same argument as the degree 3 case applies, and for $B^{-1}(\lambda_1) = \{z_i\}$, $B^{-1}(\lambda_2) = \{w_i\}$ we can find a map $M : z_i \mapsto w_i$. This map is conjugate to the rotation $f : z \mapsto \mu z$ where $\mu = \omega_n^j \sqrt[n]{\lambda_2} / \sqrt[n]{\lambda_1}$, so our map M can be written as $M = (\varphi_a^\mu)^{-1} \circ f \circ \varphi_a^\mu$.

In particular, we can find an LFT $M = \varphi_b^\eta : z_i \mapsto z_{i+1}$. For example, if $f : z \mapsto \omega_n z$ we have $M = (\varphi_a^\mu)^{-1} \circ f \circ \varphi_a^\mu : z_i \mapsto z_{i+1}$.

The matrix form of this LFT is

$$\begin{pmatrix} \mu & -\mu a \\ -\bar{a} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu & -\mu a \\ -\bar{a} & 1 \end{pmatrix}$$

Importantly, this map is independent of the choice of λ for $B^{-1}(\lambda) = \{z_i\}$.

Applying the theorem above to φ_b^η , we see that the line joining z_i and $\varphi_b^\eta(z_i) = z_{i+1}$ is always tangent to the ellipse C_b^η. But this means that C_b^η is the Blaschke curve of B.

2.2.2. *Result.* Computations by Michael Gambardella yielded

$$b = a\frac{\omega_n - 1}{\omega_n - |a|^2}$$
$$\eta = \frac{\omega_n - |a|^2}{1 - \omega_n |a|^2}$$

Result:

The Blaschke curve of the Blaschke product $B(z) = \varphi_{a'}^{\mu'} \circ [z \mapsto z^n] \circ \varphi_a^\mu(z)$ is the ellipse

- with foci $a\frac{\omega_n - 1}{\omega_n - |a|^2}$ and $a\frac{1-\omega_n}{1-\omega_n|a|^2}$
- and eccentricity $|a\frac{\omega_n-1}{\omega_n-|a|^2}| = |a\frac{\omega_n-1}{1-\omega_n|a|^2}|$

where ω_n is the principal n'th root of unity.

Lastly, I can compute the length M of the major axis.

$$M = \frac{|b - -\eta b|}{e} = |\frac{(1+\omega_n)(1-|a|^2)}{1-\omega_n|a|^2}|$$

2.2.3. *Roots.* If we assume $B(0) = 0$, we can use Xinbo Li's computations to find the roots $\{r_i : i = 0, 1, 2, \cdots n-1\}$ of the Blaschke product.

$$r_i = a\frac{1 - w_n^i}{1 - w_n^i |a|^2}$$

In particular

$$r_0 = 0$$
$$r_1 = a\frac{1 - w_n}{1 - w_n|a|^2}$$
$$r_{n-1} = a\frac{1 - w_n^{-1}}{1 - w_n^{-1}|a|^2} = a\frac{w_n - 1}{w_n - |a|^2}$$

so the foci of the ellipse are roots of the Blaschke product!

2.2.4. *Extension.* The other roots of the Blaschke product also have a geometric significance. Noe that the choice of ω_n, the principal root, over some other ω_n^j was, essentially arbitrary. Choosing $j > 1$ we may obtain an LFT sending $z_i \mapsto z_{i+j}$. The lines joining z_i and z_{i+j} are then tangent to an ellipse. Again, by replacing ω_n with ω_n^j, we see the foci of this ellipse corresponds with a different pair roots of the Blaschke product (or with a single root, in which case the ellipse degenerates to a single point.

Lastly, the map between roots of unity and roots of the Blachke product is an LFT, hence the roots lies on a single circle.

2.3. **Direct proof of tangencies and converse.**

2.3.1. *Introduction.* Given two distinct complex projective conics, C and D, there are three possibilities for their manner of intersection. This is a consequence of Bézout's theorem.

(1) C and D are in general position - they intersect at 4 distinct points.
(2) C and D intersect at 3 points, one of which is a point of tangency.
(3) C and D are doubly tangent - they intersect at only 2 points, which are both points of tangency.

In the case $C = D$, the conics are tangent everywhere. For the rest of this document we will use "doubly tangent" to include this case.

We can relate this to Blaschke products. Let D be the complex projectivisation of the unit circle $x^2+y^2 = 1$. Suppose B is a degree n Blaschke product, whose Blaschke curve is an ellipse C. Let C also denote the corresponding complex projective conic to C.

(1) The conics C and D are doubly tangent, then B is of the form $B(z) = \phi\big((\varphi(z))^n \big)$.
(2) If C and D are in general position, then B is **not** of the form $B(z) = \phi\big((\varphi(z))^n \big)$.
(3) It cannot occur C and D intersect at 3 points, one of which is a point of tangency.

This will be proved at the end of this document (Theorem 8 and Lemma 14).

There is an elegant proof of Poncelet's theorem for conics in general position, due to Griffiths and Harris [**?**]. Their proof also applies to real conics, provided the corresponding complex conics are in general position. Another form of this proof was given in the Bachelor's thesis of R.T. Buring [**?**], which also assumes the conics are in general position.

There is a theorem, due to Frantz [**?**], which relates Möbius transformations to a certain class of conic.

Definition 2.1. A real conic C with foci f_1, f_2 and eccentricity e is said to be of Möbius type if

$$|f_1| = |f_2| = e,$$

where $|\cdot|$ denotes distance from the origin.

If $f_1 = f_2 = 0 = e$, it is further required that the conic, which must be a circle, has radius less than 1.

In this document, I wish to prove the following theorem connecting these results.

Theorem 2. *A real conic C, whose interior does not contain the unit circle, is of Möbius type if and only if the corresponding complex conic is doubly tangent to the unit circle.*

The proof is in three parts. First, we will parameterise the conics of Möbius type. Secondly, we will generalise this parameterisation and show all conics doubly tangent to the unit circle are thus parameterised. Lastly, we will determine when these conics have a real component.

2.3.2. *Background.* We will make use of the symmetries of the unit circle.

Lemma 1. *Given a point p on the affine variety $x^2 + y^2 = 1$, there is a linear transformation $M \in \mathbb{SO}(2)$ which maps p to $(1,0)$, maps the conic $x^2 + y^2 = 1$ to itself, and maps other conic sections to conic sections.*

Lemma 2. *Given a point $p \in \mathbb{C}$, there is at least one linear transformation $M \in \mathbb{SO}(2)$ which maps p to $(p', 0)$ for some $p' \in \mathbb{C}$, maps the conic $x^2 + y^2 = 1$ to itself, and maps other conic sections to conic sections.*

Transformations of this kind will be useful in simplifying the problem.

2.3.3. *Ellipses doubly tangent to the unit circle.* We will parameterise the ellipses doubly tangent to the unit circle in this section. We will consider two cases:

(1) Both points of tangency lie in the affine plane.
(2) At least one point of tangency lies at infinity.

2.3.3.1. Both points of tangency lie in the affine plane.

Lemma 3. *Let C be an affine complex conic that is tangent to the circle $x^2+y^2 = 1$ at two distinct points. Then C is equivalent, up to transformation by $M \in \mathbb{SO}(2,\mathbb{C})$, to a conic C' which is tangent to $x^2+y^2 = 1$ at distinct points $(X,Y),(X,-Y) \in \mathbb{C}^2$.*

Proof. Suppose C is tangent to $x^2 + y^2 = 1$ at points $p_1, p_2 \in \mathbb{C}$. Then choose a rotation M sending $\frac{p_1+p_2}{2}$ to some point $(p', 0)$, by Lemma 2. By linearity $M(p_1) + M(p_2) = (2p', 0)$ so we can write $M(p_1) = (X,Y)$ and $M(p_2) = (X,-Y)$. Further, M preserves $x^2 + y^2 = 1$, conics, and points of tangency, so $M(C)$ is tangent to $x^2 + y^2 = 1$ at (X,Y) and $(X,-Y)$. $\square$

Lemma 4. *Let C be an affine complex conic which is tangent to $x^2 + y^2 = 1$ at the distinct points $(X,Y),(X,-Y) \in \mathbb{C}^2$. Then C can be written in one of the following forms.*

$$\{(x,y) \in \mathbb{C}^2 \mid 2XAx^2 + 2XBx + (2XA + B)y^2 = 2AX + BX^2 + B\} \text{ if } X \neq 0$$

$$\{(x,y) \in \mathbb{C}^2 \mid ax^2 + y^2 = 1\} \text{ if } X = 0$$

Proof. Suppose C is written in the general form for a conic, with parameters defined up to a common non-zero complex factor.

$$Ax^2 + Bx + Cxy + Dy + Ey^2 = F$$

Substituting the two points of tangency, we can conclude that

$$CXY + DY = 0 \text{ and } AX^2 + BX + EY^2 = F.$$

Because the points of tangency are distinct, we know $Y \neq 0$ so we can further conclude that $D = -CX$. This reduces our conic to the form

$$Ax^2 + Bx + Cxy - CXy + Ey^2 = F.$$

Next we consider the tangents. The tangent to $x^2+y^2 = 1$ at (X, Y) and $(X, -Y)$ are $Xx + Yy = 1$ and $Xx - Yy = 1$ respectively. Meanwhile, the tangents to the conic C are $2AXx + B(x + X) + C(Xy + Yx) - CX(y + Y) + 2EYy = 2F$ and $2AXx + B(x + X) + C(Xy - Yx) - CX(y - Y) - 2EYy = 2F$ respectively.

Writing a line $ax + by = c$ in homogeneous co-ordinates $[a : b : c]$, we can express the fact that the tangents at each point coincide.

$$[2AX + B + CY : 2EY : 2F - BX + CXY] = [X : Y : 1]$$

$$[2AX + B - CY : -2EY : 2F - BX - CXY] = [X : -Y : 1]$$

As a consequence $2AX + B + CY = 2EX = 2AX + B - CY$, hence $CY = 0$ and $C = 0$. It also follows that $2AX + B = 2EX$ and so $B = 0$ whenever $X = 0$. We can rewrite our equation of the conic C once more.

$$Ax^2 + Bx + Ey^2 = F \tag{1}$$

Next we will consider the case $X \neq 0$. We can write $A + \frac{B}{2X} = E$, and recalling that (X, Y) is on the circle, write

$$F = AX^2 + BX + E(1 - X^2) = A + \frac{B}{2X} + \frac{BX}{2}$$

This yields the following form.

$$Ax^2 + Bx + (A + \frac{B}{2X})y^2 = A + \frac{B}{2X} + \frac{BX}{2} \tag{2}$$

$$2XAx^2 + 2XBx + (A2X + B)y^2 = 2XA + B + BX^2 \tag{3}$$

In the case $X = 0$, we know $B = 0$ hence $F = E$ and equation 2.3.3.1 becomes

$$Ax^2 + Ey^2 = E.$$

If either A or E vanishes, then the conic C is degenerate, hence both are non-zero. Thus the conic can be written in the desired form, with $a = A/E \neq 0$.

□

Combining these two lemmas, we obtain the following.

Lemma 5. *Suppose C is a complex conic that is tangent to the circle $x^2 + y^2 = 1$ at two points on the affine plane. Up to a change of co-ordinates by $M \in \mathbb{SO}(2, \mathbb{C})$, the conic C can be written in one of the two forms below.*

$$2XAx^2 + 2XBx + (2XA + B)y^2 = 2AX + BX^2 + B$$
$$ax^2 + y^2 = 1$$

2.3.3.2. A tangent point at infinity. The projective circle $x^2+y^2=z^2$ has two points at infinity, $[1:i:0]$ and $[1:-i:0]$, with tangent lines $[1:i:0]$ and $[1:-i:0]$ respectively.

The following lemma greatly simplifies this case.

Lemma 6. *A real projective conic is tangent to $x^2+y^2=z^2$ on the line $z=0$ if and only if it is a circle centred at the origin.*

Proof. Write the conic in the following general form, where all coefficients are real.

$$Ax^2+Bxz+Cxy+Dyz+Ey^2=Fz^2$$

Without loss of generality, assume the conic is incident to $[1:i:0]$. Then

$$A+Ci-E=0.$$

We then compute the tangent at this point,

$$2Ax+Bz+C(y+xi)+Diz+2(A+Ci)yi=0.$$

Comparing to the tangent to the circle, we obtain

$$[1:i:0]=[2A+Ci:2Ai-C:B+Di]$$

from which it follows that $B+Di=0$. Note that both B and D are assumed to be real, so the latter condition forces $B=D=0$. We obtain the form

$$Ax^2+Cxy+(A+Ci)y^2=Fz^2$$

Aiming for contradiction, suppose $C\neq 0$. The conic is real for the ratio A/C is real, and the ratio $(A+Ci)/C=A/C+i$ is real. This implies their difference i is real too, a contradiction. Thus $C=0$, from where it follows that $A=E$. This provides the form $Ax^2+AY^2=Fz^2$, a circle centred at the origin, so we are done.

□

Hence no real conic, that is not a circle, is tangent to the projectivisation of $x^2+y^2=1$ at infinity.

2.3.3.3. General description. We can combine Lemma 5 with the above lemma, to obtain the following theorem.

Theorem 3. *If a complex projective conic is doubly tangent to the unit circle, then its affine part has one of the following forms.*

(1) $2XAx^2+2XBx+(2XA+B)y^2=2AX+BX^2+B$ *with* $X\neq 0$.
(2) $ax^2+y^2=1$ *with* $a\neq 1$.
(3) $x^2+y^2=f$.

2.3.4. *Conics of Möbius type.* Following from Definition 2.1, we note that a conic with distinct foci is uniquely defined by it's eccentricity. However, if the foci coincide then the conic has eccentricity 0, hence both foci are at $(0,0$ and the conic is a circle about the origin, whose radius may vary.

Lemma 7. *If C is a conic of Möbius type with foci $f_1=f_2\in\mathbb{R}^2$ then C can be written algebraically as*

$$x^2+y^2=R^2$$

for some non-zero real R.

We will simplify our analysis using rotations.

Lemma 8. *Suppose C is a conic of Möbius type. Then there is another conic C', of Möbius type, with foci $(P,Q),(P,-Q) \in \mathbb{R}^2$, and a rotation $M \in \mathbb{SO}(2,\mathbb{R})$ such that $C = M(C')$.*

Proof. The identity rotation suffices if the foci coincide, so assume they are distinct.

There is a rotation about the origin sending the line joining foci to a vertical line. If the first focus of the rotated conic is (P,Q), then the second focus is (P,Q'). However, rotation about the origin preserves the norm, so $P^2+Q^2 = P^2+Q'^2$. The foci are distinct so $Q' = -Q$. ☐

Next we seek a general form for these reflected conics.

Lemma 9. *A non-degenerate conic with foci (P,Q) and $(P,-Q)$ and semi-major axis M can be written as follows.*

$$\frac{(x-P)^2}{M^2-Q^2} + \frac{y^2}{M^2} = 1$$

Proof. Without loss of generality $Q > 0$. The inter-focal length is $2Q$, so the semi-minor axis is $m = \sqrt{|M^2-Q^2|}$.

As the conic has two foci on the affine plane it is not a parabola. If it is an ellipse, $M \geq Q$ must hold, for the conic to have points. If it is an hyperbola, $M \leq Q$. Up to translation by $-(P,0)$ the conic is in standard position. Thus if it is an ellipse it has form

$$\frac{(x-P)^2}{m^2} + \frac{y^2}{M^2} = 1,$$

while if it is an hyperbola it has form

$$-\frac{(x-P)^2}{m^2} + \frac{y^2}{M^2} = 1.$$

Together with the previous remark, this proves the lemma. ☐

Theorem 4. *A non-circle conic of Möbius type can be written, up to a rotation, in the following form.*

$$\frac{(x-P)^2}{1-P^2-Q^2} + y^2 = \frac{Q^2}{P^2+Q^2}$$

Proof. The conic is not a circle, so it has distinct foci. Applying the previous two lemmas we apply a rotation and reach the form

$$\frac{(x-P)^2}{M^2-Q^2} + \frac{y^2}{M^2} = 1$$

where M is the semi-major axis length.

The conic is of Möbius type so it has eccentricity $e = \sqrt{P^2+Q^2}$. We know $Q \neq 0$ and $e = Q/M$ leading to the final form.

$$\frac{(x-P)^2}{(Q/e)^2-Q^2} + \frac{y^2}{(Q/e)^2} = 1$$

$$\frac{(x-P)^2}{1-e^2} + y^2 = \frac{Q^2}{e^2}$$

$$\frac{(x-P)^2}{1-P^2-Q^2} + y^2 = \frac{Q^2}{P^2+Q^2}$$

☐

We also define a generalised form.

Definition 2.2. Let $(P,Q) \in \mathbb{C}^2$ satisfy $Q \neq 0$ and $0 \neq P^2+Q^2 \neq 1$. A complex affine conic C is said to be an $M(P,Q)$ conic if it is given, up to a change of co-ordinates by $m \in \mathbb{SO}(2,\mathbb{C})$, by the following equation.

$$\frac{(x-P)^2}{1-P^2-Q^2} + y^2 = \frac{Q^2}{P^2+Q^2}$$

Note that all non-circle conics of Möbius type are $M(P,Q)$ conics, and that all $M(P,Q)$ conics are also $M(P,-Q)$ conics.

Lemma 10. *All $M(P,Q)$ conics are doubly tangent to $x^2+y^2=1$.*

Proof. Let C be an $M(P,Q)$ conic. In the co-ordinates given by the definition, the intersection of C and $x^2+y^2=1$ occurs at

$$-2Px + x^2\left(P^2+Q^2\right) = \frac{-P^2}{P^2+Q^2}$$

which has a single solution $x = \frac{P}{P^2+Q^2}$.

Computing tangents at (X,Y) we get

$$\frac{(x-P)(X-P)}{1-P^2-Q^2} + yY = \frac{Q^2}{P^2+Q^2}.$$

At a point of intersection with y-coordinate Y, the tangent line to C is

$$\frac{(x-P)\left(\frac{P}{P^2+Q^2}-P\right)}{1-P^2-Q^2} + yY = \frac{Q^2}{P^2+Q^2}$$

which reduces, provided the conditions on (P,Q) in Definition 2.2 are met, to $\frac{xP}{(P^2+Q^2)} + yY = 1$. Which is exactly the tangent to the circle $x^2+y^2=1$ at that point of intersection. $\square$

Theorem 5. *All conics of Möbius type are doubly tangent, as projective complex varieties, to $x^2+y^2=1$.*

Proof. Let C be our conic of Möbius type. If C is not a circle, then it is the real part of an $M(P,Q)$ conic, and so the theorem holds by the previous lemma.

Otherwise, C is a circle centred at the origin. Hence it is a dilation of $x^2+y^2=1$. Dilations about the origin preserve points at infinity. Moreover, all tangents to C at infinity are of the form $xX+yY=0$, and so are homogeneous in their affine co-ordinates. Thus the tangents are preserved under the dilation, so C is tangent (at infinity) to $x^2+y^2=1$. $\square$

2.3.5. *Correspondence between types of conic.* We have found descriptions of conics doubly tangent to the unit circle in Theorem 3. We have also described the class of $M(P,Q)$ conics, which are doubly tangent to the unit circle by Lemma 10, however, by the previous theorem, they do not account for all conics with this property.

The following lemma further connects these two types of conic.

Lemma 11. *If a complex projective conic C is doubly tangent to the unit circle, then one of the following holds.*

1. *C is an $M(P,Q)$ conic.*
2. *C is a parabola of the form $Xx+y^2=1+X^2$.*

(3) *C has the form $x^2 + y^2 = f$.*

Proof. We will appeal to the classification in Theorem 3. The third cases are identical, so we will only consider the first two cases.

First, suppose C has the form $ax^2 + y^2 = 1$. We are working over $\mathbb{C}$, so choose Q such that $a = 1/(1 - Q^2)$. If $P = 0$, then (P, Q) satisfies the conditions to define an $M(P, Q)$ conic provided $a \neq 1$. If $a = 1$ then case 3 applies.

Second, consider the case where C has the form $2XAx^2 + 2XBx + (2XA + B)y^2 = 2AX + BX^2 + B$.

Suppose $A = 0$. Then the conic becomes a parabola $2XBx + By^2 = BX^2 + B$, or $B = 0$ and it is degenerate.

If $A \neq 0$, write $b = B/A$ to obtain the form $2Xx^2 + 2Xbx + (2X + b)y^2 = 2X + bX^2 + b$. Conics of this type are parameterised by X and b where $X \neq 0$. We also note that $b = 0$ gives the equation of the unit circle, for which case 3 applies, so we may assume $b \neq 0$ also.

An $M(P, Q)$ ellipse has the expanded form $x^2(P^2 + Q^2) - 2Px(P^2 + Q^2) + y^2(1 - P^2 - Q^2)(P^2 + Q^2) = Q^2 - (P^2 + Q^2)^2$ parameterised by P and Q where $Q \neq 0$.

We seek an equivalence.

$$
\begin{aligned}
&[\ P^2 + Q^2 : 2P(P^2 + Q^2) : (1 - P^2 - Q^2)(P^2 + Q^2) : Q^2 - (P^2 + Q^2)^2\ ] \\
=&[\ 2X : 2Xb : 2X + b : 2X + bX^2 + b\ ]
\end{aligned}
$$

Using $b = 2P$ and $X = -P/(P^2 + Q^2)$, we can check the projective equivalence term by term.

$$
\begin{aligned}
\frac{P^2 + Q^2}{2X} &= \frac{P^2 + Q^2}{-2P/(P^2 + Q^2)} &&= -\frac{(P^2 + Q^2)^2}{2P} \\
\frac{2P(P^2 + Q^2)}{2Xb} &= \frac{2P(P^2 + Q^2)}{-4P^2/(P^2 + Q^2)} &&= -\frac{(P^2 + Q^2)^2}{2P} \\
\frac{(1 - P^2 - Q^2)(P^2 + Q^2)}{2X + b} &= \frac{(1 - P^2 - Q^2)(P^2 + Q^2)^2}{2P(P^2 + Q^2) - 2P} &&= -\frac{(P^2 + Q^2)^2}{2P} \\
\frac{Q^2 - (P^2 + Q^2)^2}{2X + bX^2 + b} &= \frac{Q^2 - (P^2 + Q^2)^2}{-2P/(P^2 + Q^2) + 2P^3/(P^2 + Q^2)^2 + 2P} &&= -\frac{(P^2 + Q^2)^2}{2P}
\end{aligned}
$$

It remains to check that for every (b, X) there exists a corresponding (P, Q). That is, to check if for every (b, X) with $X \neq 0 \neq b$ there exists a (P, Q) with $Q \neq 0$ such that

$$(b, X) = (2P, -\frac{P}{P^2 + Q^2}).$$

Immediately, $P = b/2 \neq 0$. Then there is some $Q \in \mathbb{C}$ for which $-\frac{b/2}{(b/2)^2 + Q^2} = X$. Finally, we have to check this pair (P, Q) satisfies the conditions of Definition 2.2.

If $Q = 0$, then $X = -2/b$ and the conic simplifies to $-(b + 2x)^2 + (b^2 - 4)y^2 = 0$ which is degenerate.

If $Q^2 + P^2 = 1$ then $X = -b/2$ and the conic simplifies to $\left(x + \frac{b}{2}\right)^2 = 0$ which is degenerate.

Lastly, the case $Q^2 + P^2 = 0$ is never required because $X \in \mathbb{C}$ and $b \neq 0$.

This completes the classification.

□

We also seek a correspondence between conics of Möbius type and $M(P,Q)$ conics. The first step is to establish when an $M(P,Q)$ conic is a real conic.

Lemma 12. *Consider an $M(P,Q)$ conic C in $\mathbb{C}^2$. If the intersection of C and the real plane $\mathbb{R}^2$ is a conic, then $P \in \mathbb{R}$ and $Q^2 \in \mathbb{R}$. Further, the change of co-ordinates is a real rotation.*

Proof. First, note $M(P,Q)$ and $M(P,-Q)$ coincide, so we may assume $Q \geq 0$ without loss of generality. Let our change of co-ordinates be given by $M \in \mathbb{SO}(2,\mathbb{C})$. Then $M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for some $a, b \in \mathbb{C}$ with $a^2 + b^2 = 1$. In our standard co-ordinates, the equation of the conic is

$$\frac{(ax+by-P)^2}{1-P^2-Q^2} + (-bx+ay)^2 = \frac{Q^2}{P^2+Q^2}$$

which we suppose defines a real conic.

We can expand and simplify, writing $R = P^2 + Q^2 \neq 0$.

$$a^2x^2+b^2y^2+P^2-2Pax-2Pby+2abxy+(b^2x^2-2abxy+a^2y^2)(1-P^2-Q^2) = \frac{Q^2}{P^2+Q^2}(1-P^2-Q^2)$$

$$x^2(1-b^2R)+y^2(1-a^2R)-2Pax+2abxyR-2Pby = 1-\frac{P^2}{R}-R$$

It remains to verify this is a real conic. That is,

$$[1-b^2R : 1-a^2R : -2Pa : 2Rab : -2Pb : 1-\frac{P^2}{R}-R] \in \mathbb{PR}^5 \subset \mathbb{PC}^6.$$

First suppose $P \neq 0$ so $[a : b] \in \mathbb{PR}$. Next suppose $a \neq 0$ so we can write $b = \mu a$ for $\mu \in \mathbb{R}$. It follows that $a^2 + b^2 = (1+\mu^2)a^2 = 1$ so $a^2 \in \mathbb{R}$ and $b^2 \in \mathbb{R}$. Likewise, if $a = 0$ then $a^2 = 0$ so $b^2 = 1$ and both squares are real.

Continuing in the case $P \neq 0$, we see that either $R = 1/a^2 \in \mathbb{R}$ or $[1-a^2R : 1-b^2R] \in \mathbb{PR}$. Choose $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_1(1-a^2R) = \mu_2(1-b^2R)$. So $\mu_1 - \mu_2 = R(a^2-b^2)$. We conclude that either $R \in \mathbb{R}$ or $a^2 - b^2 = 0$.

If $a^2 - b^2 = 0$ then $a, b = \pm 1/\sqrt{2} \in \mathbb{R}$, hence $2Rab/-2Pb = Ra/P$ is real and so $R/P \in \mathbb{R}$. We also know $\frac{1-a^2R}{-2Pa} = \mp\frac{2-R}{4P}\sqrt{2}$ so the ratio $\frac{2-R}{P}$ is also real, so $2/P \in \mathbb{R}$ and lastly $P \in \mathbb{R}$ and $R \in \mathbb{R}$.

Alternatively,$a^2 - b^2 \neq= 0$ and so $R \in \mathbb{R}$. Then both Pa and Pb must be real. We know $a^2 + b^2 = 1$ and each individual square is real, so one square must be positive. Hence at most one of a or b is imaginary and the other is real. However, Pa and Pb are real so all three values P, a, b are real.

Lastly, we consider the case $P = 0$ where we must only guarantee

$$[1-b^2R : 1-a^2R : 2Rab : 1-R] \in \mathbb{PR}^3 \subset \mathbb{PC}^4.$$

In the case $R \neq 1$ we can find real ratios $\frac{1-b^2R}{1-R}$ and $\frac{1-a^2R}{1-R}$ whose sum $\frac{2-R}{1-R}$ is also real. It follows that $R \in \mathbb{R}$. Hence a^2, b^2, ab are real. It follows that $(a+b)^2 \in \mathbb{R}$ so $a+b$ is either real or imaginary. But a and b are either both real, or one is real and the other imaginary. In the latter case $a+b$ is real if the imaginary component is 0 and imaginary if the real component is 0. In either case the other component must be identically 1, so we are done.

In the case $R = 1$ we must guarantee

$$[1 - b^2 : 1 - a^2 : 2ab] = [a^2 : b^2 : 2ab] \in \mathbb{PR}^2 \subset \mathbb{PC}^3,$$

but this requires $[ab : b^2] \in \mathbb{PR}$ and $[ab : a^2] \in \mathbb{PR}$ from which it follows that $a, b \in \mathbb{R}$.

Thus we have shown that a, b, P, R are always real. It follows that $Q^2 = R - P^2$ is real also.

It follows that $M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{SO}(2,\mathbb{R}) \subset \mathbb{SO}(2,\mathbb{C})$. □

There are two further cases to consider, the case where Q is purely real and the case where Q is purely imaginary. We would like to be able to distinguish them.

Lemma 13. *If $(P,Q) \in \mathbb{R}^2$ satisfy the conditions of Definition 2.2 then all $M(P,Q)$ conics C are conics of Möbius type.*

Proof. Choose points $(P,Q), (P,-Q) \in \mathbb{R}^2$. As $Q \neq 0$ they are distinct. Then the conic D of eccentricity $\sqrt{P^2 + Q^2}$ is uniquely defined, and is an $M(P,Q)$ conic by Lemma 9 and Theorem 4. The conic C is equivalent to D up to a change of coordinates by $M = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{SO}(2,\mathbb{C})$. To show that C is a conic of Möbius type, we must show that a and b are real and hence M is simply a rotation.

We write the equation for C and it's expansion with $R = P^2 + Q^2$.

$$\frac{(ax + by - P)^2}{1 - P^2 - Q^2} + (-bx + ay)^2 = \frac{Q^2}{P^2 + Q^2}$$

$$x^2\left(1 - b^2 R\right) + y^2\left(1 - a^2 R\right) - 2Pax - 2Pby + 2abRxy = \frac{R - P^2 - R^2}{R}$$

An identical analysis to the previous lemma shows $a, b \in \mathbb{R}$.

□

Definition 2.3. The interior of a conic section C, given as the intersection of a cone surface with the plane, is the intersection of the solid cone with the plane.

Theorem 6. *Let a real conic C be an $M(P,Q)$ conic (hence P, Q^2 are real). Then the circle $x^2 + y^2 = 1$ lies within the interior of C when $Q^2 < 0$ and lies without the interior of C when $Q^2 > 0$.*

Proof. If $Q^2 > 0$ then $Q \in \mathbb{R}$ and so C is a conic of Möbius type. It follows that the unit circle lies without the interior of C.

Alternatively, let $-Q^2 = S > 0$. The conic C is, up to a rotation, given by the following equation.

$$\frac{(x - P)^2}{1 + S - P^2} + y^2 = \frac{S}{S - P^2}$$

We will consider separately the cases where C is alternatively an ellipse or an hyperbola.

The conic C is an ellipse exactly when $S - P^2 > 0$, hence $1 + S - P^2 > 0$. In this case the interior is given by

$$\frac{(x - P)^2}{1 + S - P^2} + y^2 \leq \frac{S}{S - P^2}$$

so

$$(x - P)^2 + y^2(1 + S - P^2) - \frac{S}{S - P^2}(1 + S - P^2) \leq 0.$$

For a point (x, y) on the unit circle, the expression on the left above simplifies to

$$-\frac{\left(\left(S-P^2\right)x+P\right)^2}{S-P^2}$$

which is always negative. Thus the circle lies within the interior of C.

The conic C is a conic exactly when $1+S-P^2<0$, hence $S-P^2<0$. In this case, the interior is given by

$$\frac{(x-P)^2}{1+S-P^2}+y^2 \le \frac{S}{S-P^2}$$

so

$$(x-P)^2+y^2\left(1+S-P^2\right)-\frac{S}{S-P^2}\left(1+S-P^2\right)\ge 0.$$

For a point (x, y) on the unit circle, the expression on the left above simplifies to

$$-\frac{\left(x\left(S-P^2\right)+P\right)^2}{S-P^2}$$

which is always positive. Thus the circle lies within the interior of C.

□

2.3.6. *Connection to Blaschke products.* We shall first prove a converse to Theorem 5 from which Theorem 2 can be derived.

Theorem 7. *If a real conic C is doubly tangent to the unit circle, and does not contain the unit circle in it's interior, then it is a conic of Möbius type.*

Proof. Appealing to the classification in Lemma 11, one of the following holds.

(1) C is an $M(P,Q)$ conic.
(2) C is a parabola of the form $Xx+y^2=1+X^2$.
(3) C has the form $x^2+y^2=f$.

Any real parabola, which is doubly tangent to the unit circle, must contain the unit circle, so we can discount the second case.

Any (non-degenerate) real conic of the form $x^2+y^2=f$ must have $f>0$. If $0<f<1$ then the circle is of Möbius type. If $1<f$ then the unit circle is contained in the interior of C, and so we can discount it.

Lastly, any real $M(P,Q)$ conic must have $P, Q^2 \in \mathbb{R}$ by Lemma 12. If $Q^2>0$ then C is a conic of Möbius type, by Lemma 13. If $Q^2<0$, then the unit circle lies within the interior of C, by the previous theorem. Thus we can discount the case $Q^2<0$. By definition $Q\neq 0$.

This completes the proof.

□

Theorem 8. *A Blaschke product B is of the form h, with ϕ, φ linear fractional transformations, if and only if the corresponding Blaschke curve C is doubly tangent to the unit circle as a complex projective variety.*

Proof. We will first prove the forward direction. If the Blaschke product is of this form, then the Blaschke curve is of Möbius type. Hence it is doubly tangent to the unit circle by Theorem 5.

Note that all Blaschke curves are contained in the unit circle. Hence the unit circle is not contained in their interior. For the converse, suppose that C is doubly tangent to the unit circle (note that we cannot count tangents which occur at a singular point). Applying Bezout's theorem, C is a degree 2 curve. The unit circle contains C , hence C is an ellipse. By the previous theorem, C is an ellipse of Möbius type. Thus choose the LFT φ corresponding to C as described in Frantz's paper [?]. If the Blaschke product is of order n then we must have $\varphi^n = \mathrm{Id}$.

This allows us to write, for some LFT θ in matrix form.

$$\varphi = \theta^{-1} \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & 1 \end{pmatrix} \theta$$

This follows from diagonalisation of φ. The diagonalising matrix θ must fix the unit circle. Without loss of generality, θ is a Blaschke factor as if θ is not a Blaschke factor then $1/\theta$ is.

The Blaschke product $B'(z) = \theta(z)^n$ will then have C as it's Blaschke curve. By Theorem 11.12 of the Finding Ellipses textbook [?], there is an LFT ϕ such that $B(z) = \phi \circ B'(z) = \phi\big((\theta(z))^n \big)$.

□

Lemma 14. *If C is the Blaschke curve of a Blaschke product, then C cannot intersect the curve $x^2 + y^2 = 1$ at two distinct points and lie tangent at a third point.*

Proof. Assuming C has no singularities, we can apply Bezout's theorem to conclude C is a degree 2 curve, a conic.

Complex conjugation is a field automorphism of $\mathbb{C}$, hence conics are invariant under the map $(x, y) \mapsto (\bar{x}, \bar{y})$. Further, this map preserves tangents. If C and the unit circle are tangent at (x, y), they must also be tangent at $(\bar{x}, \bar{y})$. By assumption they are not doubly tangent, so $(\bar{x}, \bar{y}) = (x, y)$. Hence both co-ordinates are real. Thus C intersects the real unit circle. However, no Blaschke curve intersects the unit circle, other then the curve corresponding to $B(z) = z$. □

2.4. **Approach via Collineation.**

Lemma 15. *Suppose $\phi(z) = \frac{z-a}{1-\bar{a}z}$ is a Blaschke factor. If $z_1, z_2, z_3, w_1, w_2, w_3$ are points on $\mathbb{T}$ such that the three lines $z_i \leftrightarrow w_i$ are concurrent, then the three lines $\phi(z_i) \leftrightarrow \phi(w_i)$ are concurrent.*

Proof. We will move to projective co-ordinates, so points $x + iy$ correspond to co-ordinates $[x, y, 1]^T$. Given a pair $x + iy, t + is$ we can then compute the projective co-ordinates of the line joining the points, as a cross product

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} t \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} y - s \\ t - x \\ xs - ty \end{bmatrix}.$$

Suppose $z_n = x_n + iy_n$ and $w_n = t_n + is_n$, then the three lines $z_i \leftrightarrow w_i$ are concurrent exactly when the following determinant vanishes.

$$\det \begin{bmatrix} y_1 - s_1 & y_2 - s_2 & y_3 - s_3 \\ t_1 - x_1 & t_2 - x_2 & t_3 - x_3 \\ x_1 s_1 - t_1 y_1 & x_2 s_2 - t_2 y_2 & x_3 s_3 - t_3 y_3 \end{bmatrix} = \det M$$

We now seek to find projective co-ordinates for the points $\phi(z_i), \phi(w_i)$. Suppose $a = a_R + ia_I$.

$$\begin{aligned}\phi(x+iy) &= \frac{-ia_I - a_R + x + iy}{-(x+iy)(-ia_I + a_R) + 1} \\ &= \frac{-a_I^2 x + 2a_I a_R y + a_R^2 x - a_R x^2 - a_R y^2 - a_R + x}{a_I^2 x^2 + a_I^2 y^2 - 2a_I y + a_R^2 x^2 + a_R^2 y^2 - 2a_R x + 1} \\ &+ i\frac{a_I^2 y + 2a_I a_R x - a_I x^2 - a_I y^2 - a_I - a_R^2 y + y}{a_I^2 x^2 + a_I^2 y^2 - 2a_I y + a_R^2 x^2 + a_R^2 y^2 - 2a_R x + 1}\end{aligned}$$

Thus we can give projective co-ordinates to $\phi(x+iy)$.

$$\begin{bmatrix} -a_I^2 x + 2a_I a_R y + a_R^2 x - a_R x^2 - a_R y^2 - a_R + x \\ a_I^2 y + 2a_I a_R x - a_I x^2 - a_I y^2 - a_I - a_R^2 y + y \\ a_I^2 x^2 + a_I^2 y^2 - 2a_I y + a_R^2 x^2 + a_R^2 y^2 - 2a_R x + 1 \end{bmatrix}$$

In the case that $x+iy$ is on the unit circle, $x^2+y^2 = 1$, this simplifies.

$$\begin{bmatrix} -a_I^2 x + 2a_I a_R y + a_R^2 x - 2a_R + x \\ a_I^2 y + 2a_I a_R x - 2a_I - a_R^2 y + y \\ a_I^2 - 2a_I y + a_R^2 - 2a_R x + 1 \end{bmatrix}$$

We can compute the line joining $\phi(x+iy)$ and $\phi(t+is)$, where $x+iy$ and $t+is$ are on the unit circle.

$$\begin{aligned}&\begin{bmatrix} -a_I^2 x + 2a_I a_R y + a_R^2 x - 2a_R + x \\ a_I^2 y + 2a_I a_R x - 2a_I - a_R^2 y + y \\ a_I^2 - 2a_I y + a_R^2 - 2a_R x + 1 \end{bmatrix} \times \begin{bmatrix} -a_I^2 t + 2a_I a_R s + a_R^2 t - 2a_R + t \\ a_I^2 s + 2a_I a_R t - 2a_I - a_R^2 s + s \\ a_I^2 - 2a_I s + a_R^2 - 2a_R t + 1 \end{bmatrix} \\ = (1 - a_R^2 - a_I^2) &\begin{bmatrix} a_I^2 s - a_I^2 y + 2a_I a_R t - 2a_I a_R x - a_R^2 s + a_R^2 y + 2a_R sx - 2a_R ty - s + y \\ a_I^2 t - a_I^2 x - 2a_I a_R s + 2a_I a_R y + 2a_I sx - 2a_I ty - a_R^2 t + a_R^2 x + t - x \\ a_I^2 sx - a_I^2 ty + 2a_I t - 2a_I x + a_R^2 sx - a_R^2 ty - 2a_R s + 2a_R y + sx - ty \end{bmatrix}\end{aligned}$$

The initial factor is non-zero, because Blaschke factors require $|a| \neq 1$. Because we are working in projective co-ordinates, we can ignore it.

We write the line $\phi(x+iy) \leftrightarrow \phi(t+is)$, where $x+iy$ and $t+is$ are on the unit circle as

$$\begin{bmatrix} A(x,y,t,s) \\ B(x,y,t,s) \\ C(x,y,t,s) \end{bmatrix}$$

We can compute the three lines $\phi(z_i) \leftrightarrow \phi(w_i)$ by this method. The three lines are concurrent exactly when the following determinant vanishes.

$$\det \begin{bmatrix} A(x_1,y_1,t_1,s_1) & A(x_2,y_2,t_2,s_2) & A(x_3,y_3,t_3,s_3) \\ B(x_1,y_1,t_1,s_1) & B(x_2,y_2,t_2,s_2) & B(x_3,y_3,t_3,s_3) \\ C(x_1,y_1,t_1,s_1) & C(x_2,y_2,t_2,s_2) & C(x_3,y_3,t_3,s_3) \end{bmatrix} = \det M'$$

Using a compute algebra system, I can calculate the following relation.

$$\frac{\det M'}{\det M} = (1 - a_R^2 - a_I^2)^3$$

This means $\det M'$ is a multiple of $\det M$ by a non-zero factor. Thus $\det M'$ vanishes exactly when $\det M$ vanishes.

It follows that the three lines $z_i \leftrightarrow w_i$ are concurrent exactly when the three lines $\phi(z_i) \leftrightarrow \phi(w_i)$ are concurrent.

In the case that $z_i = w_i$ we can interpret the line $z_i \leftrightarrow w_i$ as the tangent to $\mathbb{T}$ at $z_i = wi$. The exact same method as above can be used to show the theorem holds in this case.

□

We embed $\mathbb{T}$ and $\mathbb{D}$ in $\mathbb{PR}^2$ in the canonical way. Define $L\mathbb{T}$ to be the set of lines in $\mathbb{PR}^2$ which are incident to $\mathbb{T}$. We will define a new map $\phi^* : L\mathbb{T} \to L\mathbb{T}$ as follows. Given a line $l \in L\mathbb{T}$, it must intersect $\mathbb{T}$ in two points p_1, p_2 (or one double point $p_1 = p_2$). Define

$$\phi^*(l) = \phi(p_1) \leftrightarrow \phi(p_2).$$

This is well defined, because $\phi : \mathbb{T} \to \mathbb{T}$ and the line $a \leftrightarrow b$ is independent of the order of a, b, thus $\phi^*(l)$ is independent of the order in which we chose p_1, p_2.

On tangents to $\mathbb{T}$ at $p \in \mathbb{T}$ we have $\phi^*(p \leftrightarrow p) = \phi(p) \leftrightarrow \phi(p)$, so the map ϕ^* takes tangents to tangents.

We can rephrase the previous lemma using this definition.

Lemma 16. *If $l_1, l_2, l_3 \in L\mathbb{T}$, then l_1, l_2 and l_3 are conccurent if and only if $\phi^*(l_1), \phi^*(l_2)$ and $\phi^*(l_3)$ are concurrent.*

Theorem 9. *Suppose $\phi(z) = \frac{z-a}{1-\bar{a}z}$ is a Blaschke factor. There is a well-defined map $\Phi : \mathbb{PR}^2 \to \mathbb{PR}^2$ such that :*

- $\Phi|_{\mathbb{T}} = \phi$
- Φ *is a collineation.*

Proof. First we will define Φ. For any point $P \in \mathbb{PR}^2$ there are uncountably many lines $l \in L\mathbb{T}$ which are incident to P. Choose two such lines l_1, l_2 and define Φ as follows.

$$\Phi(P) = \phi^*(l_1) \cap \phi^*(l_2)$$

To show this is well defined, consider a different pair of lines $k_1, k_2 \in L\mathbb{T}$ intersecting at P. Then l_1, l_2, k_1 and l_1, l_2, k_2 are concurrent at P. Thus $\phi^*(l_1), \phi^*(l_2), \phi^*(k_1)$ are concurrent at $\Phi(P)$, and $\phi^*(l_1), \phi^*(l_2), \phi^*(k_2)$ are conccurent at $\Phi(P)$. Hence $\phi^*(k_1) \cap \phi^*(k_2) = \Phi(P)$.

Next we will prove the first bullet point. Consider a point $P \in \mathbb{T}$. If l_1, l_2 intersect at P, then they can be written $l_1 = P \leftrightarrow Q_1$ and $l_2 = P \leftrightarrow Q_2$ with $Q_1, Q_2 \in \mathbb{T}$.

$$\begin{aligned}\Phi(P) &= \phi^*(l_1) \cap \phi^*(l_2)\\ &= \big(\phi(P) \leftrightarrow \phi(Q_1)\big) \cap \big(\phi(P) \leftrightarrow \phi(Q_2)\big)\\ &= \phi(P)\end{aligned}$$

So the restriction of Φ to $\mathbb{T}$ is exactly the original Blaschke factor.

Finally, we will show that Φ is a collineation. Suppose $P, Q, R \in \mathbb{PR}^2$ are collinear. We will address this in two cases.

If P, Q, R lie on a line $l \in L\mathbb{T}$, we may use that line l to compute Φ. Then $\Phi(P), \Phi(Q), \Phi(R)$ are all incident to $\phi^*(l)$ by definition. This means $\Phi(P), \Phi(Q), \Phi(R)$ are also collinear.

Next suppose P, Q, R lie on a line $l \notin L\mathbb{T}$. It follows that $P, Q, R \notin \bar{\mathbb{D}}$. Each point lies on exactly two tangents to $\mathbb{T}$, and denote the six points of tangency by $P', P'', Q', Q'', R', R''$ respectively.

Let the tangent to $\mathbb{T}$ at $X \in \mathbb{T}$ be denoted by t_X. We know $\Phi(P) = \phi^*(t'_P) \cap \phi^*(t''_P) = t_{\phi(P')} \cap t_{\phi(P'')}$, with similar identities for Q and R.

The points are collinear, so the polars $P' \leftrightarrow P'', Q' \leftrightarrow Q'', R' \leftrightarrow R''$ are concurrent. It follows from the previous lemma that $\phi(P') \leftrightarrow \phi(P''), \phi(Q') \leftrightarrow \phi(Q''), \phi(R') \leftrightarrow \phi(R'')$ are concurrent. Thus the poles $t_{\phi(P')} \cap t_{\phi(P'')}, t_{\phi(Q')} \cap t_{\phi(Q'')}, t_{\phi(R')} \cap t_{\phi(R'')}$ are collinear.

That is, $\Phi(P), \Phi(Q), \Phi(R)$ are collinear.

□

By abuse of notation, we will also write Φ to refer to the dual map sending lines to lines.

Theorem 10. *Suppose $\phi(z) = \mu\frac{z-a}{1-\bar{a}z}$ is a Blaschke factor, and $B(z) = \phi(z)^n$ is a Blaschke product. Then the Blaschke curve of B is an ellipse doubly tangent to the unit circle.*

Proof. We have shown previously (in a different document) that the preimages of the Blaschke product B are of the following form,

$$B^{-1}(\lambda) = \{z_i : i = 1, 2, \cdots n\}$$
$$z_i = \phi^{-1}(\sqrt[n]{\lambda}\omega^i)$$

where ω is the principal n'th root of unity.

The polygon given by joining each $\sqrt[n]{\lambda}\omega^i$ to $\sqrt[n]{\lambda}\omega^{i+1}$ is tangent to a circle centred at the origin. Indeed, this polygon is tangent to the same circle C regardless of the value of λ.

Let Φ be the collineation corresponding to ϕ^{-1}, from the previous theorem. Then $\Phi(\sqrt[n]{\lambda}\omega^i) = z_i$, so Φ sends tangent lines to C to tangent lines to the Blaschke curve. Further, if z_i, z_{i+1} is tangent to the Blaschke curve, it must be the image of a tangent to C. This means $\Phi(C^*)$ is the dual curve of the Blaschke curve, hence $\Phi(C)$ is the Blaschke curve.

We know collineations map conics to conics, so $\Phi(C)$ is a conic. Hence the Blaschke curve is a conic, hence an ellipse.

Lastly, every real collineation extends to a complex collineation. Circles centred at the origin are always doubly tangent to the unit circle, as complex projective varieties. Hence $\Phi(C)$ is doubly tangent to $\Phi(\mathbb{T}) = \phi(\mathbb{T}) = \mathbb{T}$. □

Corollary 1. If B is a Blaschke product conjugate to z^n, then there is a collineation fixing $\mathbb{T}$ which turns the Blaschke curve of B into a circle.

Corollary 2. If ϕ is a Blaschke factor and B is a Blaschke product, then the Blaschke curves of $B \circ \phi$ and B are projectively equivalent (by a projectivity which fixes the unit circle).

Proof. Sketch of proof:

The map ϕ sends 0s of $B \circ \phi$ to 0s of B. Then ϕ extends to a collineation Φ which maps the polygon joining 0s of $B \circ \phi$ to the polygon joining 0s of B. Thus it maps envelopes to envelopes. Collineations preserve the property 'is a conic', and any conic contained in the circle is an ellipse. □

Corollary 3. The Blaschke curve of B is an ellipse if and only if the Blaschke curve of $B \circ \phi$ is an ellipse for every Blaschke factor ϕ. In fact, if we can find a single ϕ such that $B \circ \phi$ gives an ellipse, then B also gives an ellipse.