Topological K-Theory

a generalized cohomology theory via vector bundles

Big picture

Roadmap

Slogan

Topological K-theory is a reduced generalized cohomology theory built from vector bundles.

$$X \quad \rightsquigarrow \quad \mathrm{Vect}_{\mathbb{C}}(X) \quad \rightsquigarrow \quad K^0(X) = \mathrm{Vect}_{\mathbb{C}}(X)^{\mathrm{gp}} \quad \rightsquigarrow \quad \widetilde{K}^*(X)$$

Cohomology side

K-theory side

Cohomology theories

Singular cohomology

$$C_n(X;A) = \text{free } A\text{-module generated by singular } n\text{-simplices}$$

$$\cdots \to C_{n+1}(X;A) \xrightarrow{\partial} C_n(X;A) \xrightarrow{\partial} C_{n-1}(X;A) \to \cdots, \qquad \partial^2 = 0$$

$$C^n(X;A) = \mathrm{Hom}(C_n(X;\mathbb{Z}), A), \qquad H^n(X;A) = \frac{\ker(d : C^n \to C^{n+1})}{\mathrm{im}(d : C^{n-1} \to C^n)}.$$

Observe

Singular cohomology satisfies formal axioms that can be separated from the chain-level construction.

Ordinary cohomology theories

An ordinary cohomology theory assigns groups

$$h^n(X, A)$$

to CW pairs (X, A), contravariantly:

$$f : (X, A) \to (Y, B) \qquad \Longrightarrow \qquad f^* : h^n(Y, B) \to h^n(X, A).$$

Eilenberg–Steenrod axioms

homotopy invariance, exactness, excision, additivity, dimension.

Dimension axiom:

$$h^n(*) = 0 \qquad n \neq 0.$$

Why ordinary cohomology is rigid

Uniqueness theorem

For CW complexes, every ordinary cohomology theory is naturally singular cohomology with coefficients

$$G = h^0(*).$$

Thus

$$h^n(X, A) \cong H^n(X, A; G).$$

- Singular, cellular, and simplicial cohomology agree in ordinary settings.
- On smooth manifolds,

$$H^n_{\mathrm{dR}}(M) \cong H^n(M; \mathbb{R}).$$

- To get new theories, keep the formal axioms but drop dimension.

Reduced and generalized cohomology

The reduced package

Work with pointed CW complexes:

$$(X, x_0).$$

From unreduced to reduced:

$$\widetilde{h}^n(X) = \ker\big(h^n(X) \to h^n(*)\big).$$

From reduced back to unreduced:

$$h^n(X) = \widetilde{h}^n(X_+), \qquad h^n(X, A) = \widetilde{h}^n(X/A).$$

Reduced suspension:

$$\Sigma X = S^1 \wedge X, \qquad \Sigma S^n \cong S^{n+1}.$$

Reduced cohomology axioms

A reduced cohomology theory is a sequence of functors

$$\widetilde{h}^n : \mathsf{CW}_*^{op} \to \mathsf{Ab}.$$

For a cofiber sequence $A \to X \to X/A$:

$$\cdots \to \widetilde{h}^n(X/A) \to \widetilde{h}^n(X) \to \widetilde{h}^n(A) \xrightarrow{\delta} \widetilde{h}^{n+1}(X/A) \to \cdots .$$

Suspension axiom:

$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X).$$

Reduced dimension axiom, if ordinary:

$$\widetilde{h}^n(S^0) = 0 \qquad n \neq 0.$$

Generalized cohomology

Definition

generalized cohomology = cohomology theory without the dimension axiom.

So we keep

homotopy invariance, exactness, suspension, additivity

but allow

$$\widetilde{h}^n(S^0) \neq 0 \qquad \text{for } n \neq 0.$$

Examples:

K-theory, cobordism, stable cohomotopy.

Vector bundles and group completion

Vector bundles: the geometric input

A rank n complex vector bundle over X is a map

$$p : E \to X$$

such that each fiber

$$E_x = p^{-1}(x)$$

is an n-dimensional complex vector space, and locally

$$p^{-1}(U) \cong U \times \mathbb{C}^n.$$

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \mathbb{C}^n \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

Slogan

Vector bundles are locally trivial, but globally they may twist.

Vector bundles form a commutative monoid

Let

$$\mathrm{Vect}_{\mathbb{C}}(X) = \{\text{isomorphism classes of finite-rank complex vector bundles over } X\}.$$

Write an element as

$$[E].$$

Direct sum gives addition:

$$[E] + [F] = [E \oplus F], \qquad (E \oplus F)_x = E_x \oplus F_x.$$

Therefore

$$\mathrm{Vect}_{\mathbb{C}}(X)$$

is a commutative monoid.

Group completion

A commutative monoid is like an abelian group without inverses.

$$\mathbb{N} \rightsquigarrow \mathbb{Z}.$$

For a commutative monoid A, the group completion A^{gp} is universal:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\iota"'] & G \\
A^{\mathrm{gp}} \arrow[ur, dashed, "\exists!\, \overline{f}"'] &
\end{tikzcd}$$

Concrete model:

$$A^{\mathrm{gp}} = \mathbb{Z}\{[a] : a \in A\}/([a+b] = [a] + [b]).$$

Defining K-theory

Definition of $K^0(X)$

$$K^0(X) = \mathrm{Vect}_{\mathbb{C}}(X)^{\mathrm{gp}}.$$

An element of $K^0(X)$ is a virtual vector bundle:

$$[E] - [F].$$

The basic cancellation relation is:

$$[E \oplus H] - [F \oplus H] = [E] - [F].$$

Analogy

$$\mathbb{Z} = \mathbb{N}^{\mathrm{gp}}, \qquad K^0(X) = \mathrm{Vect}_{\mathbb{C}}(X)^{\mathrm{gp}}.$$

The point and reduced K^0

Over a point:

$$\mathrm{Vect}_{\mathbb{C}}(*) \cong \mathbb{N}, \qquad K^0(*) \cong \mathbb{N}^{\mathrm{gp}} \cong \mathbb{Z}.$$

For pointed X, define

$$\widetilde{K}^0(X) = \ker\big(K^0(X) \to K^0(*)\big).$$

Since $K^0(*) \cong \mathbb{Z}$, this map is virtual rank:

$$[E] - [F] \longmapsto \mathrm{rank}(E) - \mathrm{rank}(F).$$

Meaning

$$\widetilde{K}^0(X) = \{\text{virtual vector bundles of virtual rank } 0\}.$$

Functoriality and homotopy

A map

$$f : X \to Y$$

pulls bundles back:

$$E \to Y \qquad \rightsquigarrow \qquad f^*E \to X.$$

So

$$f^* : K^0(Y) \to K^0(X).$$

If $f \simeq g$, then

$$f^* = g^*.$$

Background classification theorem, for nice spaces:

$$\mathrm{Vect}_{\mathbb{C},n}(X) \cong [X, BU(n)].$$

Higher groups and Bott periodicity

Higher groups by suspension

Because we are using reduced theories, define negative degrees by suspension:

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X).$$

In particular:

$$\widetilde{K}^{-1}(X) = \widetilde{K}^0(\Sigma X), \qquad \widetilde{K}^{-2}(X) = \widetilde{K}^0(\Sigma^2 X).$$

Positive degrees would require formal desuspension:

$$\widetilde{K}^{n}(X) \overset{?}{=} \widetilde{K}^0(\Sigma^{-n} X).$$

Bott periodicity

Bott periodicity

$$\widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2 X).$$

Equivalently:

$$\widetilde{K}^{q+2}(X) \cong \widetilde{K}^q(X).$$

Thus:

$$\widetilde{K}^1(X) \cong \widetilde{K}^{-1}(X) = \widetilde{K}^0(\Sigma X), \qquad \widetilde{K}^2(X) \cong \widetilde{K}^0(X).$$

K-theory is generalized cohomology

Main theorem

$\widetilde{K}^*$ is a reduced generalized cohomology theory on pointed CW complexes.

It satisfies:

homotopy invariance, exactness, suspension, wedge additivity.

For a cofiber sequence $A \to X \to X/A$:

$$\cdots \to \widetilde{K}^q(X/A) \to \widetilde{K}^q(X) \to \widetilde{K}^q(A) \xrightarrow{\delta} \widetilde{K}^{q+1}(X/A) \to \cdots .$$

It fails the dimension axiom because K-theory is 2-periodic.

Serre–Swan and summary

Serre–Swan and algebraic K-theory

For compact Hausdorff X:

$$\mathrm{Vect}_{\mathbb{C}}(X) \simeq \mathrm{Proj}_{fg}(C(X)).$$

Algebraic K-theory starts from:

$$R \quad \rightsquigarrow \quad \mathrm{Proj}_{fg}(R) \quad \rightsquigarrow \quad K_0(R).$$

Final summary

$$X \longmapsto \mathsf{Vect}_{\mathbb{C}}(X) \longmapsto \mathsf{Vect}_{\mathbb{C}}(X)^{\mathrm{gp}} = K^0(X)$$

$$\widetilde{K}^0(X) = \ker(K^0(X) \to K^0(*))$$

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X)$$

$$\widetilde{K}^{q+2}(X) \cong \widetilde{K}^q(X)$$