

Affine Group Schemes, Hopf Algebras, Actions, Stabilizers, and Quotients in Algebraic Geometry

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Abstract

This writeup grew out of an exploratory project for Math 632 (Algebraic Geometry II) taught by Prof. Aaron Pixton at the University of Michigan. Its goal is to understand how some familiar notions from classical group theory, such as groups, subgroups, actions, stabilizers, and quotients, are translated into algebraic geometry. We begin with group schemes and their affine description by Hopf algebras, then discuss actions, stabilizer subgroup schemes, and the quotient problem for homogeneous spaces. The main focus is the affine case, where Chevalley's theorem realizes closed subgroups as stabilizers of lines in finite-dimensional representations, and hence allows quotients G/H to be constructed as orbits in projective space.

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0 Motivation

Throughout, k will be an algebraically closed field, and all schemes are of finite type over k . Affine schemes are written as $\text{Spec}(A)$, products are fibered products over k , and group schemes are group objects in the category of k -schemes.

The central question is: given an algebraic group G and an algebraic subgroup H , when does the homogeneous space G/H exist as a scheme? This is a special case of the broader quotient problem for group actions, but it is the cleanest and most classical case. We will understand that G/H should not be constructed by naively taking equivalence classes of points and hoping the result is a scheme. Rather, the quotient has to be characterized by a universal property, and

its existence must be justified. In the smooth affine case, the construction will work because affine algebraic groups are linear enough to turn subgroups into stabilizers, and the geometry of projective space will turn those stabilizers into orbits.

One technical point is that products of schemes are not products of underlying sets. For example,

$$\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \cong \mathbb{A}_k^2,$$

but the underlying set of \mathbb{A}_k^2 contains more points than just ordered pairs of closed points (it also contains generic points of irreducible closed subsets). However in our setting, for schemes over k , we do get

$$(X \times_k Y)(k) \cong X(k) \times Y(k),$$

and since k is algebraically closed and the schemes under consideration are of finite type, the closed points give the usual classical geometric picture.

1 Group schemes and affine Hopf algebras

A group is a set G equipped with multiplication, inverse, and identity data

$$m : G \times G \rightarrow G, \quad i : G \rightarrow G, \quad e \in G,$$

satisfying associativity, identity, and inverse axioms. The important observation is that these axioms can be expressed entirely in terms of maps and commutative diagrams. Associativity says that the two maps $G \times G \times G \rightarrow G$ given by $(g_1, g_2, g_3) \mapsto (g_1 g_2) g_3$ and $(g_1, g_2, g_3) \mapsto g_1 (g_2 g_3)$ agree. The identity and inverse axioms can similarly be expressed diagrammatically.

1.0.1 Definition (Group object). Let \mathcal{C} be a category with finite products and terminal object $*$. A *group object* in \mathcal{C} is an object G together with morphisms

$$m : G \times G \rightarrow G, \quad i : G \rightarrow G, \quad e : * \rightarrow G$$

such that the usual associativity, identity, and inverse diagrams commute.

An ordinary group is exactly a group object in the category of sets. A group scheme is obtained by making the same definition in the category of schemes.

1.0.2 Definition (Group scheme over k). A *group scheme over k* is a group object in the category of k -schemes. Equivalently, it is a k -scheme together with morphisms

$$m : G \times_k G \rightarrow G, \quad i : G \rightarrow G, \quad e : \text{Spec } k \rightarrow G$$

satisfying the associativity, identity, and inverse diagrams

$$\begin{array}{ccc}
 G \times_k G \times_k G & \xrightarrow{m \times \text{id}_G} & G \times_k G \\
 \text{id}_G \times m \downarrow & & \downarrow m \\
 G \times_k G & \xrightarrow{m} & G
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \text{Spec } k \times_k G & \xrightarrow{e \times \text{id}_G} & G \times_k G & \xleftarrow{\text{id}_G \times e} & G \times_k \text{Spec } k \\
 & \searrow \cong & \downarrow m & & \swarrow \cong \\
 & & G & &
 \end{array}$$

and, writing $u : G \rightarrow \text{Spec } k$ for the structure morphism,

$$\begin{array}{ccccc}
 G & \xrightarrow{(i, \text{id}_G)} & G \times_k G & \xleftarrow{(\text{id}_G, i)} & G \\
 & \searrow e \circ u & \downarrow m & \swarrow e \circ u & \\
 & & G & &
 \end{array}$$

1.0.3 Definition (Algebraic group). An *algebraic group over k* is a group scheme of finite type over k .

The multiplication law is now a morphism of schemes, not just on its set of points. Note the scheme structure is not exhausted by the visible k -points, but over an algebraically closed field and in finite type, the k -point picture remains a useful guide. The affine case is especially important, as we have a direct dictionary here. If $G = \text{Spec}(A)$, then morphisms of G correspond contravariantly to ring maps of A , imposing some structure on A itself.

1.0.4 Definition (Affine algebraic group). An *affine algebraic group over k* is an algebraic group G whose underlying k -scheme is affine.

Suppose $G = \text{Spec}(A)$ is an affine group scheme. The multiplication map $m : G \times G \rightarrow G$ becomes, after reversing arrows, a k -algebra map

$$\Delta : A \rightarrow A \otimes_k A,$$

called the *comultiplication*. The identity $e : \text{Spec } k \rightarrow G$ becomes a k -algebra map

$$\varepsilon : A \rightarrow k,$$

called the *counit*. The inverse $i : G \rightarrow G$ becomes a k -algebra map

$$S : A \rightarrow A,$$

called the *antipode*. Note that the coalgebra structure and algebra structure are not independent; there is some compatibility. The compatibility is encoded by requiring $\Delta : A \rightarrow A \otimes_k A$ and $\varepsilon : A \rightarrow k$ to be k -algebra maps. Equivalently, one can require the multiplication $\mu : A \otimes_k A \rightarrow A$ and unit $\eta : k \rightarrow A$ to be coalgebra maps. This leads us to a natural definition.

1.0.5 Definition (Hopf algebra). A *Hopf algebra over k* is a k -algebra A equipped with k -algebra maps

$$\Delta : A \rightarrow A \otimes_k A, \quad \varepsilon : A \rightarrow k, \quad S : A \rightarrow A$$

such that

- (i) Δ is coassociative

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes_k A \\
 \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\
 A \otimes_k A & \xrightarrow{\text{id} \otimes \Delta} & A \otimes_k A \otimes_k A
 \end{array}$$

(ii) ε is a counit

$$\begin{array}{c}
 A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\varepsilon \otimes \text{id}} k \otimes_k A \cong A \\
 \searrow \text{id} \nearrow \\
 A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\text{id} \otimes \varepsilon} A \otimes_k k \cong A
 \end{array}$$

(iii) S is an antipode

$$\begin{array}{c}
 A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{S \otimes \text{id}} A \otimes_k A \xrightarrow{m_A} A \\
 \searrow \eta \circ \varepsilon \nearrow \\
 A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\text{id} \otimes S} A \otimes_k A \xrightarrow{m_A} A
 \end{array}$$

1.0.6 Theorem. *Let $G = \text{Spec}(A)$ be affine over k . Then giving G the structure of a group scheme over k is equivalent to giving A the structure of a commutative Hopf algebra over k .*

This is the primary reason the affine case is manageable. Our basic examples are \mathbb{G}_a , \mathbb{G}_m , and GL_n . The additive group is $\mathbb{G}_a = \text{Spec}(k[t])$, with group law induced by addition. Its k -points are $(k, +)$. The coordinate function t on the target, pulled back along multiplication, becomes the function $(a, b) \mapsto a + b$ on the product. Therefore

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \quad \varepsilon(t) = 0, \quad S(t) = -t.$$

The multiplicative group is $\mathbb{G}_m = \text{Spec}(k[t, t^{-1}])$, with group law induced by multiplication on k^\times . Pulling back the coordinate t along multiplication gives the function $(a, b) \mapsto ab$, so

$$\Delta(t) = t \otimes t, \quad \varepsilon(t) = 1, \quad S(t) = t^{-1}.$$

The general linear group is

$$\text{GL}_n = \text{Spec}(k[x_{ij}, 1/\det]).$$

Here x_{ij} are the coordinate functions on the space of all $n \times n$ matrices, and inverting the determinant cuts out the open locus of invertible matrices. If $A = (a_{ij})$ and $B = (b_{ij})$, then $(AB)_{ij} = \sum_r a_{ir} b_{rj}$ and we have a formula for comultiplication

$$\Delta(x_{ij}) = \sum_r x_{ir} \otimes x_{rj}.$$

This is the prototype affine algebraic group, and it is the natural ambient group in which many closed subgroup schemes live.

1.0.7 Definition (Closed subgroup scheme). *Let G be a group scheme over k . A closed subgroup scheme of G is a closed subscheme $H \hookrightarrow G$ such that the inclusion is a homomorphism of group schemes.*

In classical group theory, a subgroup $H \leq G$ is a subset closed under multiplication and inverse. In scheme theory, the correct analogue is not merely a subset of points, but a subgroup scheme $H \hookrightarrow G$. In the affine case, this again becomes algebra. If $G = \text{Spec}(A)$, then a closed subgroup scheme corresponds to a quotient of A whose kernel is compatible with the Hopf algebra structure, namely a Hopf ideal.

2 Actions, stabilizers, and quotients

2.0.1 Definition (Action of a group scheme). Let G be a group scheme over k , and let X be a k -scheme. An *action* of G on X is a morphism

$$a : G \times_k X \rightarrow X$$

such that

(i)

$$a \circ (m \times \text{id}_X) = a \circ (\text{id}_G \times a) : G \times_k G \times_k X \rightarrow X,$$

(ii)

$$a \circ (e \times \text{id}_X) = \text{id}_X : \text{Spec } k \times_k X \cong X \rightarrow X.$$

This is the scheme-theoretic analogue of the usual definition of a group action. In ordinary group theory, an action is a map $a : G \times X \rightarrow X$ satisfying $e \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. The scheme-theoretic definition has the same formal structure, with sets and functions replaced by schemes and morphisms.

Fix a point $o \in X(k)$. The *orbit map* of o is the morphism

$$\mu_o : G \rightarrow X, \quad g \mapsto g \cdot o.$$

The *stabilizer subgroup scheme* of o , also called the *isotropy group* of o , is the fiber product

$$\begin{array}{ccc} G_o & \longrightarrow & G \\ \downarrow & & \downarrow \mu_o \\ \text{Spec } k & \xrightarrow{o} & X. \end{array}$$

Equivalently, G_o is the equalizer of the orbit map $\mu_o : G \rightarrow X$ and the constant map $c_o : G \rightarrow X$ with value o . Set-theoretically, this is the familiar stabilizer

$$G_o = \{g \in G \mid g \cdot o = o\}.$$

Scheme-theoretically, the stabilizer is the subgroup scheme cut out by the same fixing condition, interpreted inside the category of schemes. One can also speak about fixed points of an action. The fixed-point subscheme X^G , when it exists, is the largest subscheme $Y \subseteq X$ such that the restricted action $G \times_k Y \rightarrow X$ agrees with the second projection $G \times_k Y \rightarrow Y \hookrightarrow X$. This expresses the same idea; a point is fixed if every group element acts on it trivially, but the condition must be imposed scheme-theoretically.

Now suppose $H \subseteq G$ is an algebraic subgroup. In ordinary group theory, G/H is the set of right cosets gH , equivalently the quotient of G by the relation $g \sim g'$ if and only if $g^{-1}g' \in H$. More conceptually, G/H is universal among maps out of G that are constant on right H -cosets. This universal property is the correct idea to carry into algebraic geometry.

2.0.2 Definition (Quotient map). Let G be an algebraic group and $H \subseteq G$ an algebraic subgroup. A morphism

$$q : G \rightarrow Q$$

is called a *quotient map for G/H* if

- (i) $q(gh) = q(g)$ in the appropriate functorial sense, so q is constant on right H -cosets;
- (ii) for every k -scheme Y and every morphism $f : G \rightarrow Y$ constant on right H -cosets, there exists a unique morphism $\bar{f} : Q \rightarrow Y$ such that

$$f = \bar{f} \circ q.$$

In that case, Q is called a *quotient* of G by H , and is denoted G/H .

The relation between stabilizers and quotients is made precise by the following orbit-quotient criterion, in the form we use from Milne [Mil17].

2.0.3 Theorem (Orbit-quotient criterion). *Let G act on a separated algebraic scheme X , and let $o \in X(k)$. Then X , together with the orbit map*

$$\mu_o : G \rightarrow X,$$

is the quotient of G by the stabilizer G_o if and only if μ_o is faithfully flat.

Set-theoretically, surjectivity is what makes a map quotient-like. In algebraic geometry, the right strengthening is faithful flatness. Thus the quotient problem can be approached by producing a candidate orbit inside some scheme X , proving that the orbit map is faithfully flat, and then identifying that orbit with the quotient.

3 Existence of homogeneous spaces in the affine case

Projective space enters the existence proof because a representation of an algebraic group on a vector space naturally induces an action on the space of lines. If G acts linearly on a finite-dimensional vector space V , then it acts on the set of one-dimensional subspaces of V , since linear automorphisms send lines to lines. The algebro-geometric space of lines in V is the projective space $\mathbb{P}(V)$, so a representation of G on V induces an action

$$G \times \mathbb{P}(V) \rightarrow \mathbb{P}(V).$$

3.0.1 Definition (Linear action). Let G be an affine algebraic group over k , and let X be an algebraic scheme over k . An action of G on X is said to be *linear* if there exists a finite-dimensional representation

$$r : G \rightarrow \mathrm{GL}(V)$$

and an equivariant immersion

$$X \hookrightarrow \mathbb{P}(V).$$

The key representation-theoretic input is Chevalley's theorem.

3.0.2 Theorem (Chevalley). *Let G be an affine algebraic group over k , and let $H \subseteq G$ be a closed algebraic subgroup. Then there exists a finite-dimensional representation of G on a vector space V and a line $L \subseteq V$ such that*

$$H = \text{Stab}_G(L).$$

This theorem turns the subgroup H into a geometric fixing condition. Ordinary group theory already suggests that subgroups often arise as stabilizers of structures. **Theorem 3.0.2** says that, for affine algebraic groups, every closed algebraic subgroup arises in this way from a finite-dimensional representation. This is where we use the affine hypothesis. The quotient theorem then follows by placing this stabilizer condition inside projective space.

The orbit theorem we need first is due to Milne [Mil17, Proposition 7.17].

3.0.3 Proposition (Orbit immersion theorem). *Let G be a smooth algebraic group acting on a separated algebraic scheme X . For every $o \in X(k)$, the quotient G/G_o exists, and the canonical map*

$$G/G_o \rightarrow X$$

is an immersion.

This proposition says that once a smooth group acts on a separated ambient scheme, the orbit through a point is not merely a set; it is an algebraic scheme, and it sits inside X as an immersed subscheme.

Milne's existence theorem for homogeneous spaces is the following consequence [Mil17, Theorem 7.18].

3.0.4 Theorem (Existence of homogeneous spaces). *If G is smooth and affine, then for every closed algebraic subgroup $H \subseteq G$, the quotient G/H exists as a separated algebraic scheme.*

Proof. Let $H \subseteq G$ be a closed algebraic subgroup. By **Theorem 3.0.2**, there is a finite-dimensional representation

$$\rho : G \rightarrow \text{GL}(V)$$

and a line $L \subseteq V$ such that

$$H = \text{Stab}_G(L).$$

The representation induces an action of G on projective space,

$$G \times \mathbb{P}(V) \rightarrow \mathbb{P}(V),$$

because every linear automorphism of V sends lines to lines. Let $o = [L] \in \mathbb{P}(V)(k)$ be the point corresponding to L . By construction, $G_o = H$.

Applying **Proposition 3.0.3** to the action of the smooth group G on the separated scheme $\mathbb{P}(V)$, the quotient G/G_o exists, and the canonical map

$$G/G_o \rightarrow \mathbb{P}(V)$$

is an immersion. Since $G_o = H$, this gives G/H as a separated algebraic scheme, realized geometrically as the orbit of $[L]$ in projective space. \square

The proof can be summarized as follows. The subgroup H is realized as the stabilizer of a line, the line gives a point of projective space, and the quotient G/H becomes the orbit of that point. So,

$$H \subseteq G \implies H = \text{Stab}_G(L) \implies G/H = G \cdot [L] \subseteq \mathbb{P}(V).$$

There is also a non-smooth extension of the orbit statement [Mil17, Proposition 7.20].

3.0.5 Proposition (Orbit theorem). *Let G act on a separated algebraic scheme X . For every $o \in X(k)$, the quotient G/G_o exists, and the canonical map*

$$G/G_o \rightarrow X$$

is an immersion.

This removes the smoothness hypothesis from [Proposition 3.0.3](#). For the main quotient story, however, the smooth affine case gives the cleanest geometric proof and the clearest conceptual picture.

4 $\mathrm{GL}_2/H \cong \mathbb{P}^1$

One example is the identification

$$\mathrm{GL}_2/H \cong \mathbb{P}^1.$$

Let $G = \mathrm{GL}_2$ act on k^2 in the usual way, and let $L = \langle e_1 \rangle$ be the line spanned by the first basis vector. The stabilizer of L is the subgroup

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in k^\times, b \in k \right\}.$$

Indeed, a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

preserves L if and only if it sends e_1 into the span of e_1 . But

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

and this vector lies in the span of e_1 exactly when $c = 0$. Thus the stabilizer is precisely H . Since GL_2 acts transitively on the set of one-dimensional subspaces of k^2 , and the space of such lines is \mathbb{P}^1 , the orbit of L is all of \mathbb{P}^1 . [Theorem 3.0.4](#) gives

$$\mathrm{GL}_2/H \cong \mathbb{P}^1.$$

Note that a quotient of an affine algebraic group need not be affine (and usually is not).

4.0.1 Remark. We can also place the affine case inside the broader structure theory of algebraic groups. The discussion above focuses on the affine direction: affineness gives Hopf algebras and linear representations, and those in turn construct homogeneous spaces G/H . But algebraic groups are not all affine. The Barsotti–Chevalley theorem says that a connected algebraic group contains a connected affine normal subgroup N such that G/N is an abelian variety.

References

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